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Tunneling Radiation of a Torus-Like Black Hole in Anti-de Sitter Space-Time

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An extension of the Parikh-Wilczek's semi-classical quantum tunneling method, the tunneling radiation of the charged particle from a torus-like black hole is investigated. Difference from the uncharged mass-less particle, the geodesics of the charged massive particle tunneling from the black hole is not light-like, but determined by the phase velocity. The derived result shows that the tunneling rate depends on the emitted particle's energy and electric charge, and takes the same functional form as uncharged particle. It proves also that the exact emission spectrum is not strictly pure thermal, but is consistent with the underlying unitary theory.

KEY WORDS: a torus-like black hole; energy conservation; charge conservation; tunneling rate; Bekenstein-Hawking entropy.

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1. INTRODUCTION

Hawking radiation is viewed as a tunneling process and the radiation spectrum is pure thermal one supposing the space-time background of the black hole is fixed (Hawking, 1975, 2005). Recently, a method to describe Hawking radiation as tunneling process, where a particle moves in dynamic geometry, has been developed by Kraus and Wilczek (1995a,b) and Keski-Vakkuri (1996) and utilized by Parikh and Wilczek with considerable success (Parikh and Wiltzek, 2000; Parikh, 2002, 2004), who carried out research on the tunneling radiation characteristics of static spherically symmetric Schwarzchild black hole and Reissner-Nordström black hole. The results display that the derived radiation spectrum is not strictly thermal under the consideration of energy conservation and the unfixed space-time

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background. Following that, Hemming and Keski-Vakkuri have investigated the Hawking radiation from Anti-de Sitter black hole (Hemming and Keski-Vakkuri, 2001) and Medved has researched those from a de Sitter cosmological horizon Medved (2002). The results support the Parikh-Wilczek's opinion. In 2005, Zhang and Zhao have extended the Parikh-Wilczek's semi-classical quantum tunneling method to discuss the tunneling radiation characteristics of the axi-symmetric black hole (Zhang and Zhao, 2005a,b) and got a satisfying result. Besides above mentioned cases, a lot of works have also been carried out for further development of this method (Arzano, 2006; Arzano *et al.*, 2005; Fang *et al.*, 2005; Jiang *et al.*, 2006a,b; Jiang and Wu, 2006a,b; Li *et al.*, 2005, 2006; Medved and Vagenas, 2005a,b; Qi *et al.*, 2006; Ren *et al.*, 2005; Setare and Vagenas, 2005; Yang *et al.*, 2005, 2006; Vagenas, 2001, 2002a,b). All of them indicate that the true Hawking radiate spectra are not pure thermal when the self-gravitation is taken into account.

In order to extend our understanding on quantum tunnel effect of the black hole, we attempt to investigate the tunneling radiation of the charged particle from the event horizon of a torus-like black hole in Anti-de Sitter space. In fact, since the massive particles are not light-like particles, they do not follow the light-like geodesics in radial when tunneling across the horizon. Meanwhile, a general black hole should be charged and the corresponding tunneling particle should be also charged, so effect of the electro-magnetic field of the emitted particles should not be ignored. For a charged black hole, not only should the energy conservation but also the electric charge conservation be considered (Kraus and Wilczek, 1995c; Zhang and Zhao, 2005c,d). Two particularly significant point of this paper are as follows: On one hand, we need to find the equation of motion of a charged massive tunneling particle. We can treat the massive charged particle as de Broglie wave, then the motion equation was obtain by calculating the phase velocity of de Broglie wave corresponding to the outgoing particle. On the other hand, we should also consider the effect of the electro-magnetic field outside the black hole when a charged particle tunnels out. As the Lagrangian function of the electro-magnetic field corresponding to the generalized coordinates described by A_μ is $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, which are ignorable coordinates, we can modify the lagrangian function by eliminating the freedoms corresponding to these coordinates. Thus, the emission rate for a charged massive particle is derived with revised Lagrangian function and semi-classical approximation.

2. COORDINATE TRANSFORMATION AND MOTION EQUATION

The line element of a torus-like black hole in Anti-de Sitter space-time with sign $(-, +, +, +)$ can be written as (Huang and Liang, 1995)

$$ds^2 = -\left(-\frac{1}{3}\Lambda r^2 - \frac{2m}{\pi r} + \frac{4Q^2}{\pi r^2}\right) dt_A^2 + \left(-\frac{1}{3}\Lambda r^2 - \frac{2m}{\pi r} + \frac{4Q^2}{\pi r^2}\right)^{-1} \times dr^2 + r^2\left(d\varphi^2 + d\psi^2\right), \tag{1}$$

where, $0 \leq \varphi \leq 2\pi$ $0 \leq \psi \leq 2\pi$, t_A is the coordinate time of the black hole and Λ is the cosmological constant. And the corresponding non vanishing component of the electromagnetic vector potential may be taken to be

$$A_t = -\frac{4Q}{r}. \tag{2}$$

Obviously, above metric exits coordinate singularities at the horizon $r_\pm$ for which

$$-\frac{1}{3}\Lambda r_\pm^2 - \frac{2m}{\pi r_\pm} + \frac{4Q^2}{\pi r_\pm^2} = 0, \tag{3}$$

where r_+ is black hole event horizon which is the outermost value of r in expression (3). With tunneling radiation of the black hole in mind, we now adopt a semi-classical method based on Parikh-Wilczek framework. In order to keep the regularity at the event horizon and the stationarity of the metric which in turn implies that the time direction is a Killing vector, we perform Painlevé coordinate transformation (Painlevé, 1921) as follows

$$dt_A = dt + f(r,\psi)\,dr + g(r,\psi)\,d\psi, \tag{4}$$

where $f(r,\psi)$, $g(r,\psi)$ are two functions to be determined about r, ψ, and satisfy the integrability condition $\partial_\psi f(r,\psi) = \partial_r g(r,\psi)$. Substituting Equation (4) into Equation (1) and setting $g_{00} = -\frac{1}{3}\Lambda r^2 - \frac{2m}{\pi r} + \frac{4Q^2}{\pi r^2}$, we have

$$\begin{aligned} ds^2 = &- g_{00}dt^2 - 2g_{00}f(r,\psi)\,dtdr + \left[g_{00}^{-1} - g_{00}f^2(r,\psi)\right]dr^2 \\ &- 2g_{00}g(r,\psi)dtd\psi - 2g_{00}f(r,\psi)g(r,\psi)drd\psi \\ &+ \left[r^2 - g_{00}g^2(r,\psi)\right]d\psi^2 + r^2 d\varphi^2. \end{aligned} \tag{5}$$

By analogue to the asymptotically flat black holes, the Anti-de Sitter Painlevé coordinates should have the property (Hemming and Keski-Vakkuri, 2001) that constant time slices of the the Anti-de Sitter black hole metric (5) will have the same geometry as constant time slices of the metric (1), namely

$$g_{00}^{-1} - g_{00}f^2(r,\psi) = \left(-\frac{1}{3}\Lambda r^2\right)^{-1}, \tag{6}$$

This implies that

$$f(r,\psi) = \frac{1}{g_{00}}\sqrt{1 + \left(-\frac{1}{3}\Lambda r^2\right)^{-1} g_{00}} \tag{7}$$

So the Painlevé-torus-like metric reads off as follows

$$
\begin{aligned}
ds^2 = & -g_{00}dt^2 + 2\sqrt{1+\left(\frac{1}{3}\Lambda r^2\right)^{-1} g_{00}}dtdr - \left(\frac{1}{3}\Lambda r^2\right)^{-1} dr^2 \\
& - 2g_{00}g(r,\psi)dtd\psi + 2\sqrt{1+\left(\frac{1}{3}\Lambda r^2\right)^{-1} g_{00}}g(r,\psi)drd\psi \\
& + \left[r^2 - g_{00}g^2(r,\psi)\right]d\psi^2 + r^2 d\varphi^2,
\end{aligned} \tag{8}
$$

According to Landau's condition of coordinate clock synchronization (Painlevé, 1921), we can get the same expression as integrability condition $\partial_\psi f(r,\psi) = \partial_r g(r,\psi)$, so the Painlevé- torus-like line element (8) satisfies the condition of coordinate clock synchronization. Apart from that, the new line element has many other attractive features: Firstly, the components of either the metric or the inverse metric are well behaved at the event horizon; Secondly, the event horizon and the infinite red-shift surface are coincident with each other; Finally, the new space-time line element is stationary. Those characteristics would provide superior condition to the quantum tunneling radiation.

Now, let us derive the motion equation of the charged massive particles, which is different from that of the uncharged mass-less outgoing particles that follow the radial null geodesics

$$
\dot{r} = \sqrt{\left(\frac{1}{3}\Lambda r^2\right)^2 + \left(\frac{1}{3}\Lambda r^2\right)g_{00}} - \left(\frac{1}{3}\Lambda r^2\right), \tag{9}
$$

Because the trajectory of a charged particle is subject to Lorentz forces, the trajectory isn't the radially light-like geodesics when it tunnels across the horizon. For the sake of simplicity, the trajectory is approximately determined by the phase velocity. Here, we treat the charged particle as a de Broglie wave (Zhang and Zhao, 2005c,d). According to de Broglie's hypothesis and the definition of the phase (group) velocity, the outgoing particle that can be considered as a massive shell corresponds to a kind of 's-wave'whose phase velocity v_p and group velocity v_g have the following relationship

$$
v_p = \frac{1}{2}v_g; \qquad v_p = \frac{dr}{dt}; \qquad v_g = \frac{dr_c}{dt}, \tag{10}
$$

where r_c denotes the radial position the particle. Since the tunneling process across the barrier is an instantaneous effect, there are two events that take place simultaneously in different places during the process. One is the particle tunneling into the barrier, another is the particle tunneling out the barrier. Because the metric (8) satisfies Landau's condition of the coordinate clock synchronization (Landau and Lifshitz, 1987), the coordinate time difference of these two events is

$$dt = -\frac{g_{tr}}{g_{tt}} dr_c, \quad (d\theta = 0), \tag{11}$$

So we can easily get the group velocity

$$v_g = \frac{dr_c}{dt} = -\frac{g_{tt}}{g_{tr}} = \frac{g_{00}}{\sqrt{1 + \left(\frac{1}{3}\Lambda r^2\right)^{-1} g_{00}}}. \tag{12}$$

Therefore the phase velocity (the radial geodesics) can be expressed as

$$\dot{r} = v_p = \frac{1}{2} v_g = \frac{1}{2}\left(-\frac{g_{tt}}{g_{tr}}\right) = \frac{g_{00}}{2\sqrt{1 + \left(\frac{1}{3}\Lambda r^2\right)^{-1} g_{00}}} \tag{13}$$

In the following section, we will focus on discussing the Hawking radiation of the charged massive particle via tunneling from the black hole with the above equation of motion.

3. CHARGED MASSIVE PARTICLE'S TUNNELING RADIATION

When the particle's self-gravitation is taken into account, the line element and motion equation should be modified. According to the energy conservation and the charge conservation, we fix the total energy and the charge of the whole space-time and allow those of the black hole to change. When a particle with energy ω and charge q tunnels out from the event horizon, the black hole mass and charge will become $M - \omega$, $Q - q$. Thus the line element of the black hole is expressed as

$$\begin{aligned} ds^2 = &- g'_{00} dt^2 + 2\sqrt{1 + \left(\frac{1}{3}\Lambda r^2\right)^{-1} g'_{00}} dt dr - \left(\frac{1}{3}\Lambda r^2\right)^{-1} dr^2 \\ &- 2 g'_{00} g(r, \psi) dt d\psi + 2\sqrt{1 + \left(\frac{1}{3}\Lambda r^2\right)^{-1} g'_{00}} g(r, \psi) dr d\psi \\ &+ [r^2 - g'_{00} g^2(r, \psi)] d\psi^2 + r^2 d\varphi^2, \end{aligned} \tag{14}$$

where $g'_{00} = -\frac{1}{3}\Lambda r^2 - \frac{2(m-\omega)}{\pi r} + \frac{4(Q-q)^2}{\pi r^2}$, So the radial geodesics of the charged massive particles is

$$\dot{r} = \frac{g'_{00}}{2\sqrt{1 + \left(\frac{1}{3}\Lambda r^2\right)^{-1} g'_{00}}} \tag{15}$$

and the non-zero component of electromagnetic potential becomes

$$A_t = -\frac{4(Q-q)}{r}. \tag{16}$$

When the charged particle tunnels out, the effect of the electro-magnetic field should be taken into account. So the matter-gravity system consists of the black hole and the electromagnetic field outsides the black hole. As the Lagrangian function of the electro-magnetic field corresponding to the generalized coordinates described by A_μ is $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, we can find that the generalized coordinate $A_\mu = (A_t, 0, 0, 0)$ is an ignorable coordinate. In order to eliminate the freedom degree corresponding to A_μ, the imaginary part of the action for the charged massive particle should be written as

$$\begin{aligned} ImS &= Im \int_{t_i}^{t_f} (L - P_{A_t}\dot{A}_t)dt = Im \int_{r_i}^{r_f} (P_r\dot{r} - P_{A_t}\dot{A}_t)\frac{dr}{\dot{r}} \\ &= Im \int_{r_i}^{r_f} \int_{(0,0)}^{(P_r,P_{A_t})} \left(\dot{r}dP'_r - \dot{A}_t dP'_{A_t}\right)\frac{dr}{\dot{r}}, \end{aligned} \tag{17}$$

where r_i and r_f represent the locations of the event horizon before and after the particle with energy ω and charge q runnels out. P_r and P_{A_t} are the canonical momentums conjugate to r and A_t, respectively. According to Hamilton's canonical equation of motion, we have

$$\begin{aligned} \dot{r} &= \frac{dH}{dP_r}|_{(r;A_t,P_{A_t})}, \qquad dH|_{(r;A_t,P_{A_t})} = d(M-\omega), \\ \dot{A}_t &= \frac{dH}{dP_{A_t}}|_{(A_t;r,P_r)}, \qquad dH|_{(A_t;r,P_r)} = \frac{4(Q-q)}{r}d(Q-q). \end{aligned} \tag{18}$$

Equation(18) represents the energy changes of the black hole because of the loss of mass and charge when a particle tunnels out. By keeping the mass M and the electric charge Q fixed, the conservation of energy and electric charge will be enforced in a natural way. Substituting Equations (15), (16) and (18) into Equation (17) we obtain

$$\begin{aligned} ImS &= Im \int_{r_i(M,Q)}^{r_f(M-\omega,Q-q)} \int_{(M,Q)}^{(M-\omega,Q-q)} \left(dH|_{(r;A_t,P_{A_t})} - dH|_{(A_t;r,P_r)}\right)\frac{dr}{\dot{r}} \\ &= Im \int_{r_i(M,Q)}^{r_f(M-\omega,Q-q)} \int_{(M,Q)}^{(M-\omega,Q-q)} \frac{2\sqrt{1+\left(\frac{1}{3}\Lambda r^2\right)^{-1} g'_{00}}}{g'_{00}} \\ &\quad \times \left[d(M-\omega') - \frac{4(Q-q')}{r}d(Q-q')\right]dr. \end{aligned} \tag{19}$$

Since $g'_{00} = -\frac{1}{3}\Lambda r^2 - \frac{2(m-\omega')}{\pi r} + \frac{4(Q-q')^2}{\pi r^2} = (r - r'_+)(r - r'_-) = 0$ satisfies the horizon equation after the particle with energy ω and charge q tunnels out, there exists a single pole in Equation (19). The integral can be evaluated by deforming the contour around the pole, so as to ensure that positive energy solution decay in time, and we get

$$ImS = -Im \int_{r_i(M,Q)}^{r_f(M-\omega,Q-q)} i\pi^2 r dr = -\frac{1}{2}\pi^2(r_f^2 - r_i^2), \tag{20}$$

where $r_i = r_+(m, Q^2)$, $r_f = r'_+\left[m - \omega, (Q - q)^2\right]$. According to semi-classical WKB approximation, the relationship between the tunneling rate and the action is (Kraus and Keski-Vakkuri, 1997; Kraus and Parentani, 2000)

$$\Gamma \sim e^{-2ImS}. \tag{21}$$

And the entropy of the torus-like black hole (Han *et al.*, 2003) is

$$S_{BH} = \pi^2 r_+^2 \tag{22}$$

Therefore the tunneling rate is

$$\Gamma \sim e^{-2ImS} = e^{\pi^2[(r'^2_+) - r_+^2]} = e^{\Delta S_{BH}}, \tag{23}$$

Where $\Delta S_{BH} = S_{BH}(M - \omega, Q - q) - S_{BH}(M, Q)$ is the difference of Bekenstein-Hawking entropies of the torus-like black hole before and after the particle emission. From Equation (22), we can see that the exact emission spectrum deviates from the pure spectrum, and depends on not only the emitting particle's mass, but also electric charge. On the other hand, the result is consistent with an underlying unitary theory. As the tunneling rate in quantum mechanics is obtained by

$$\Gamma(i \to f) \sim |A_{fi}|^2 \bullet \texttt{(phase space factor)} \tag{24}$$

where $|A_{fi}|^2$ is the square of the amplitude for the tunneling action. The phase space factor is derived by the number of the initial states and averaging the number of the final states, and the number of the initial and final states are the exponent of the initial and final entropies, hence

$$\Gamma \sim \frac{e^{S_{\text{final}}}}{e^{S_{\text{initial}}}} = \exp(\Delta S) \tag{25}$$

Obviously, Equation (25) are consistent with our result obtained by applying the Parikh-Wilczek's semi-classical quantum tunneling method. So, Equation (23) satisfy the underlying unitary theory in quantum mechanics. This may provide implications for the black hole information puzzle.

4. DISCUSSION AND CONCLUSION

In this paper, we have presented the tunneling radiation of charged particles from a torus-like black hole. In order to calculate the emission rate of the black hole by Parikh-Wilczek framework, we first introduce a simple but useful Painlevé coordinate which is well behaved at the event horizon. Secondly, since the radial null geodesics is only applied to radiation of uncharged massless particles, we treat the charged massive particle as a de Brogile wave, then, the expression of the equation of motion was derived by computing the phase velocity of the de Brogile wave. Thirdly, considering effect of electromagnetic field outside the black hole when the charged particles tunnels out, electric charge conservation should be implemented. Thus, by fixing the total mass and total electric charge of the space-time but allowing those of the black hole to vary, we derive the emission rate for a charged massive particle with revised Lagrangian function and WKB approximation.

To summarize, by treating the background space-time as dynamical, the energy conservation and the electric charge conservation are enforced in a nature way when the particle's self-gravitation is taken into account. The result shows that tunneling rate is related to the change of Bekenstein-Hawking entropy and depends on the emitted particle's energy and electric charge. Meanwhile the result supports Parikh-Wilczek's conclusion, that is, the Hawking thermal radiation actually deviates from perfect thermality and is consistent with an underlying unitary theory.

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